Imperial College London

Lecture 4

Frequency Domain Analysis and Fourier Transform

Peter Y K Cheung

Dyson School of Design Engineering
Imperial College London

URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

Relationship between exponentials and sinusoids

- Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

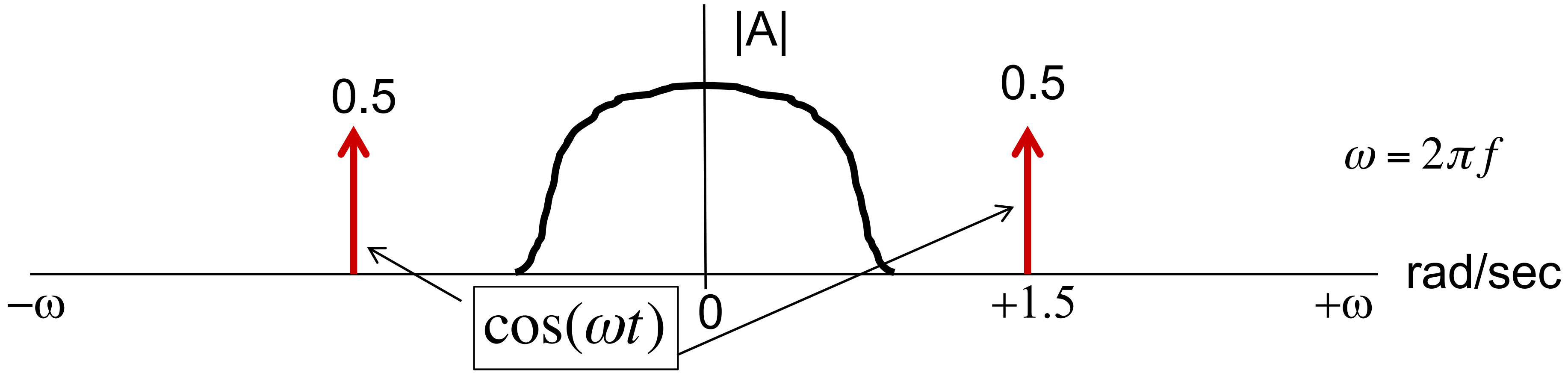
$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$
$$= \cos(\omega t) - j\sin(\omega t)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

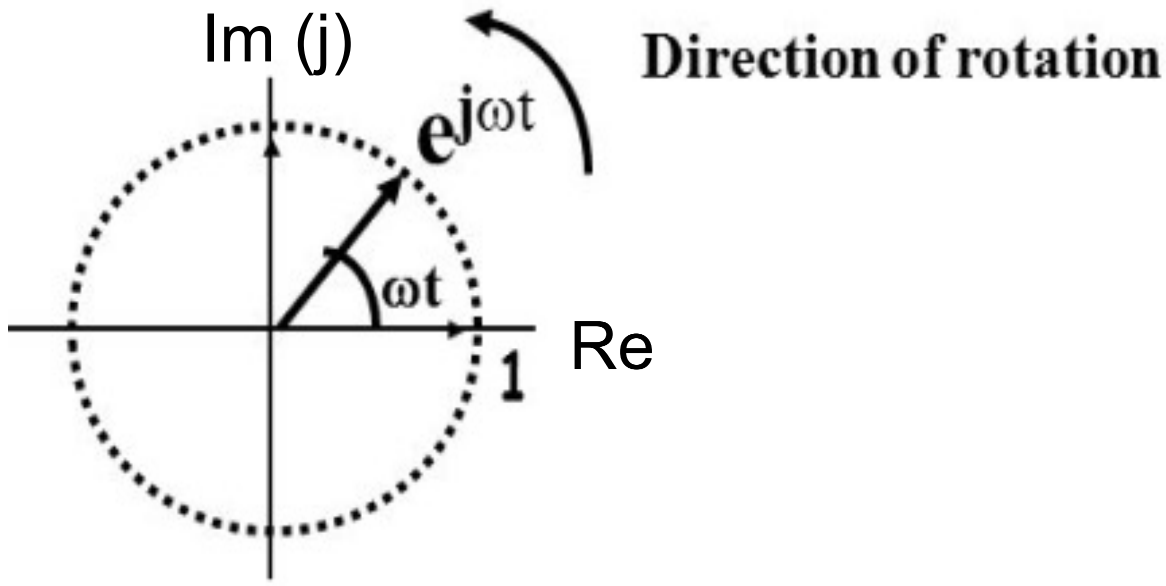
- Therefore, in signal analysis, we usual regard "frequency" to be ω in the exponential vector $e^{j\omega t}$.

- The frequency spectrum is therefore a plot of the amplitude (and phase) projected onto exponential components $e^{j\omega t}$ for different ω.

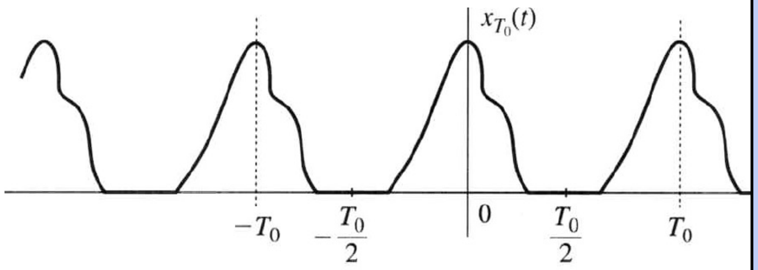


$e^{j\omega t}$ viewed as a VECTOR

- $e^{j\omega t}$ is the building block, each at different frequency ω.
- Can be viewed as a VECTOR as show below.
- The magnitude of the vector $|e^{j\omega t}|$ is 1.
- This vector is rotating in a complex plane at a rate of ω rads/sec in the direction shown.
- cos(ωt) and sin(ωt) are just the projection of the this vector on the REAL (x-axis) and IMAGINERY (y-axis) axes in this diagram.



Fourier Series in three forms



$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ )$$

$$a_n = \frac{2}{T_0}\int_0^{T_o} x(t) \cos n\omega_0 t\, dt \qquad b_n = \frac{2}{T_0}\int_0^{T_o} x(t) \sin n\omega_0 t\, dt$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

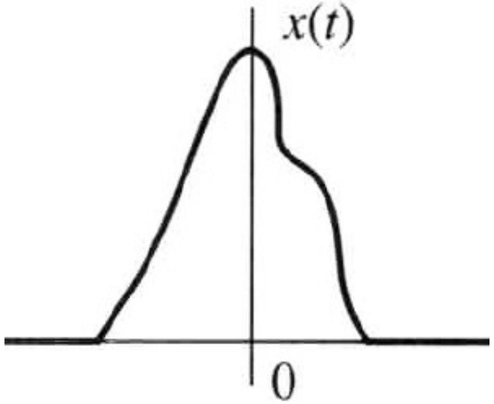
$$\theta_n = tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$x(t) \ = \ \sum_{-\infty}^{\infty}\ D_n\, e^{j(n\omega_0 t + \,\theta_n)}$$

$$D_n = \frac{1}{T_0}\int_{-T_{o/2}}^{T_{o/2}} x(t) e^{-jn\omega_0 t} dt$$

Definition of Fourier Transform

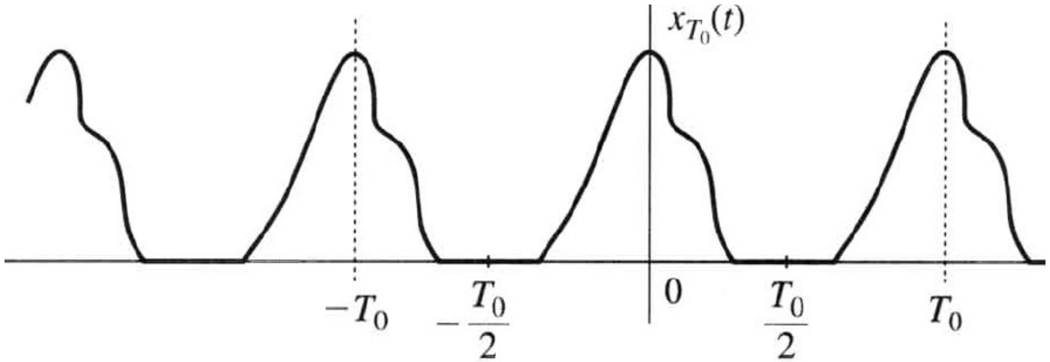
- The forward and inverse **Fourier Transform** are defined for **aperiodic** signal as:



$$X(\omega) = \mathcal{F}[x(t)] \quad = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] \quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- **Fourier series** is used for **periodic** signals.



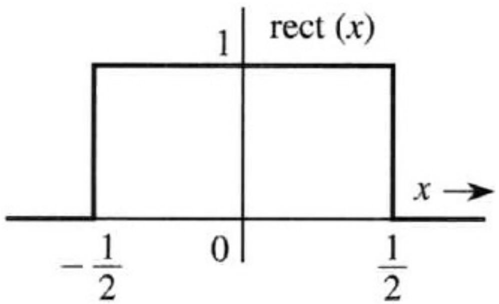
$$x(t) \ = \ \sum_{-\infty}^{\infty} \ D_n \, e^{j(n\omega_0 t + \theta_n)}$$

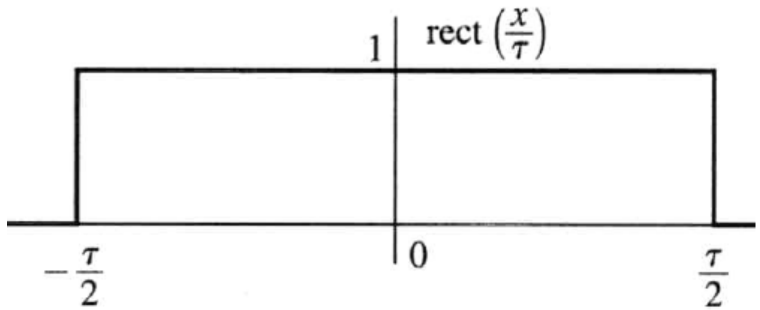
$$D_n = \frac{1}{T_0} \int_{-T_{o/2}}^{T_{o/2}} x(t) e^{-jn\omega_0 t} dt$$

Define three useful functions

- A unit rectangular window function **rect(x)**:

$$\mathrm{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

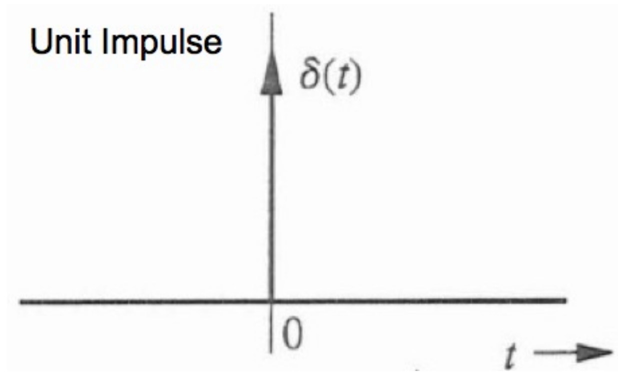




- The unit impulse function **δ(t)** (Dirac impulse):

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

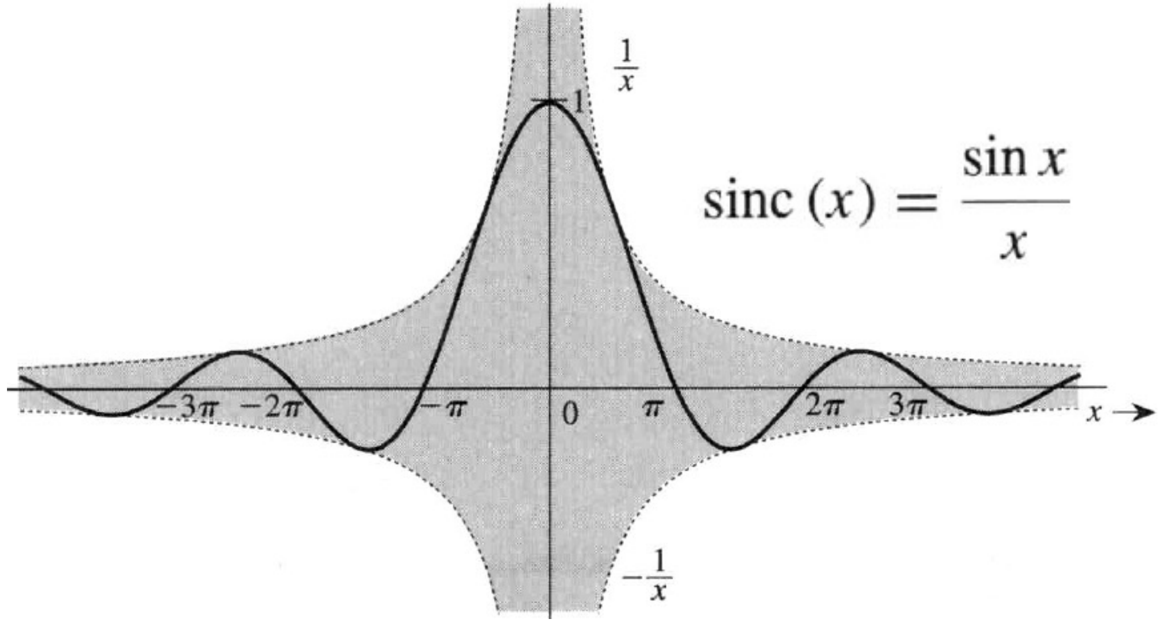


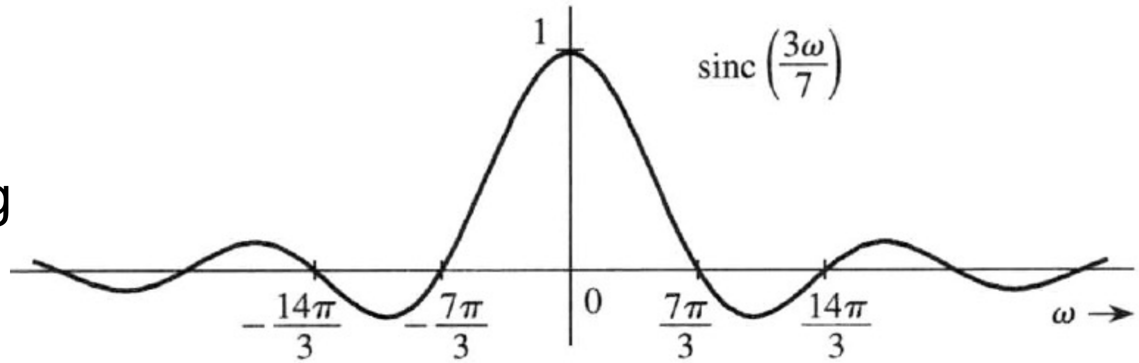
- Interpolation function **sinc(x):**

$$\mathrm{sinc}\,(x) = \frac{\sin x}{x} \qquad \text{or} \qquad \mathrm{sinc}\,(x) = \frac{\sin \pi x}{\pi x}$$

More about sinc(x) function

- **sinc(x)** is an even function of x.
- **sinc(x)** = 0 when sin(x) = 0 except when x=0, i.e. x = $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$…..
- **sinc(0)** = 1 (derived with L'Hôpital's rule)
- **sinc(x)** is the product of an oscillating signal sin(x) and a monotonically decreasing function 1/x. Therefore it is a damping oscillation with period of 2π with amplitude decreasing as 1/x.

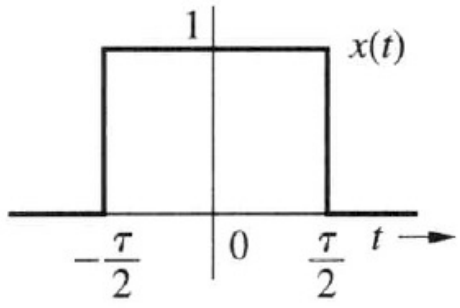




Fourier Transform of $x(t) = \text{rect}(t/\tau)$

- Evaluation:

$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

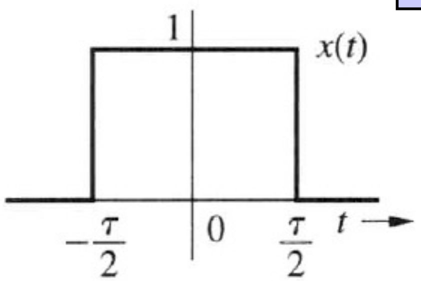


- Since rect(t/τ) = 1 for -τ/2 < t < τ/2 and 0 otherwise

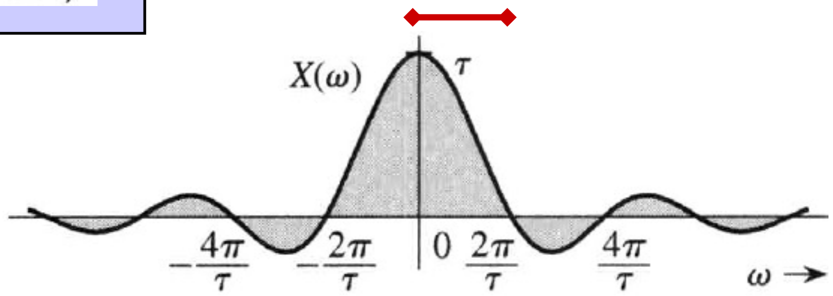
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\boxed{\text{rect}\left(\frac{t}{\tau}\right) \overset{\text{FT}}{\Longleftrightarrow} \tau\ \text{sinc}\left(\frac{\omega\tau}{2}\right)}$$

Bandwidth ≈ 2π/τ



FT
$\Longleftrightarrow$



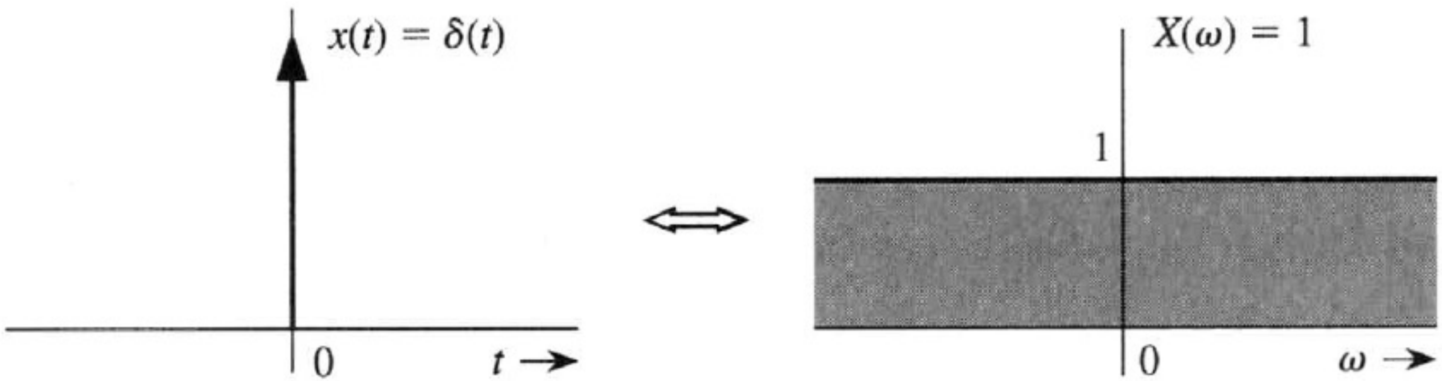
Fourier Transform of unit impulse $x(t) = \delta(t)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

- IMPORTANT – Unit impulse contains COMPONENT AT EVERY FREQUENCY.

$$\delta(t) \Longleftrightarrow 1$$



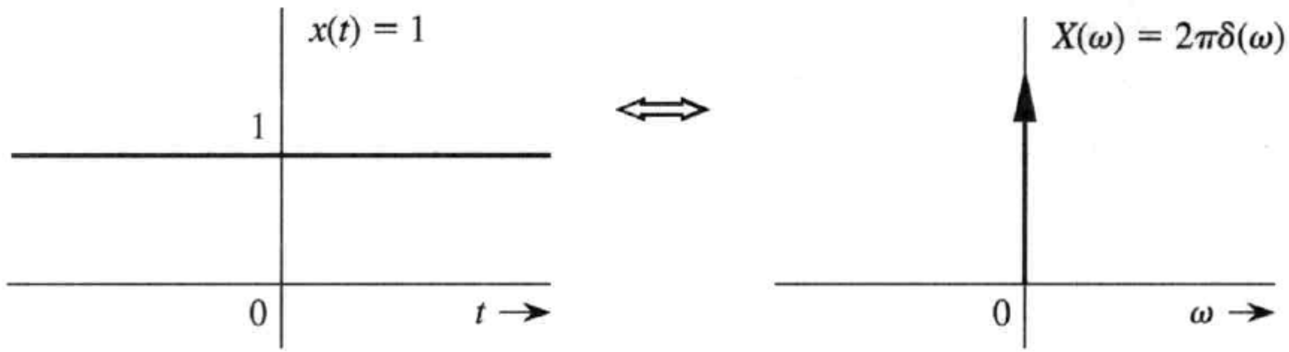
Inverse Fourier Transform of $\delta(\omega)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}$$

- Spectrum of a constant (i.e. d.c.) signal x(t) = 1 is an impulse $2\pi\delta(\omega)$.

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \qquad \text{or} \qquad 1 \Longleftrightarrow 2\pi\delta(\omega)$$



Inverse Fourier Transform of $\delta(\omega - \omega_0)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega \;\; = \frac{1}{2\pi} e^{j\omega_0 t}$$

- Spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega=\omega_0$.

$$\frac{1}{2\pi} e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\,\delta(\omega - \omega_0)$$

and

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\,\delta(\omega + \omega_0)$$

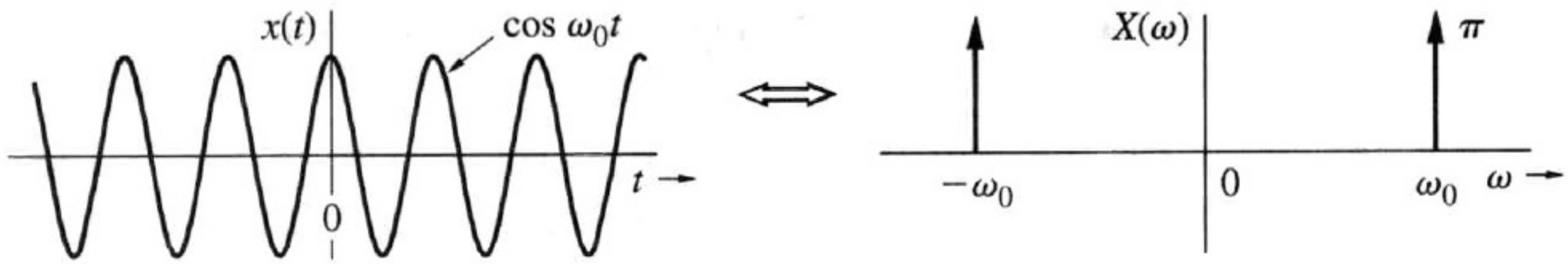
Fourier Transform of everlasting sinusoid cos $\omega_0 t$

- Remember Euler's formula: $\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

- Use results from previous slide, we get:

$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

- Spectrum of cosine signal has two impulses at positive and negative frequencies.



Fourier Transform of any periodic signal

- Fourier series of a periodic signal x(t) with period T_0 is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

- Take Fourier transform of both sides, we get:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

- This is rather obvious!

Fourier Transform of a unit impulse train

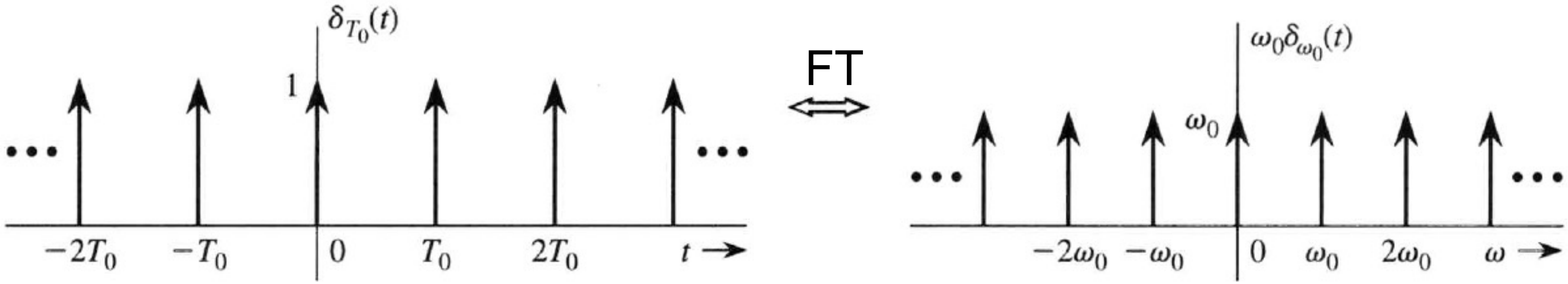
- Consider an impulse train $$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta(t - nT_0)$$
- The Fourier series of this impulse train can be shown to be:

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0} \quad \text{and} \quad D_n = \frac{1}{T_0}$$

- Therefore using results from the last slide (slide 13), we get:

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \omega_0 \delta_{\omega_0}(\omega)$$



Fourier Transform Table (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	

Fourier Transform Table (2)

No.	$x(t)$	$X(\omega)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \dfrac{j\omega}{\omega_0^2 - \omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \dfrac{\omega_0}{\omega_0^2 - \omega^2}$	
15	$e^{-at} \sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
16	$e^{-at} \cos \omega_0 t\, u(t)$	$\dfrac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$

Fourier Transform Table (3)

No.	$x(t)$	$X(\omega)$	
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\text{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\,\text{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\text{sinc}\,(Wt)$	$\text{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\text{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\text{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\displaystyle\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	

Three Big Ideas

1. **Euler formula** provides an alternative way to represent sine and cosine functions in terms of $e^{j\omega t}$ and $e^{-j\omega t}$.

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

2. **Extracting a portion of a signal** x(t) for $-\tau/2 \leq t \leq \tau/2$ can be modelled by multiplying x(t) by the rectangular function rect(x/ τ).

3. The Fourier Transform of an infinite train of unit impulses is again an infinite train of unit impulses.